The Weighted Residual Finite Elements Applied to a 1D Heat Problem

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Abstract

The fast development of computers made numerical solutions of engineering problems very much achievable using many different numerical approaches**.** Finite element method is one of the numerical techniques capable of solving problems with complex geometries. In this paper an overview of the method is presented and applied to a chosen heat transfer problem comparing the numerical results to analytical solution for illustration purposes. As shown in the results the finite elements solution is in a very good agreement to the analytical solution for the chosen problem.

الملخص

أدى التطور السريع لأجهزة الحاسوب إلى جعل الحلول العددية للمشكلات والمسائل الهندسية قابلة للتحقيق إلى حد كبير باستخدام العديد من الأساليب العددية المختلفة. طريقة العناصر المحدودة هي إحدى التقنيات العددية القادرة على حل المشكلات ذات الأشكال الهندسية المعقدة. في هذه الورقة ، يتم تقديم نظرة عامة على الطريقة وتطبيقها على مشكلة انتقال الحرارة بمقارنة النتائج العددية بالحل النظري. كما هو موضح في النتائج ، فإن حل العناصر المحدودة في اتفاق جيد جدًا مع الحل النظري للمشكلة المختارة.

Introductions

Most if not all partial differential equations that describe real engineering problems cannot be solved using analytical techniques. Instead, solutions can be approximated using numerical methods.

Finite elements is a numerical technique being applied for the solutions of differential equations in different fields of science and engineering.

Using different types of elements makes the method very much capable of solving problems with complex geometries.
The method is applied to solve problems in structural mechanics, civil engineering, fluid mechanics, heat transfer and others.
Using triangular elements as shown in figure (1) gives the finite element method the capability to cover very well the problem domain including those with irregular boundaries. The method can apply one type of element or mixed types such as rectangular and triangular in the 2D cases for instance.

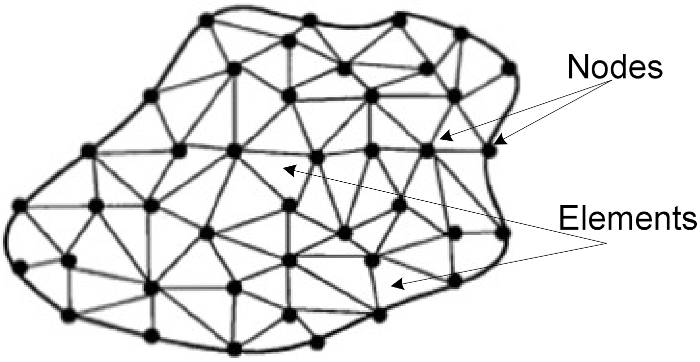


Figure (1) Example of 2D triangular elements and nodes

Theory Behind the Finite Elements Method

Weighted residual method is one of the approaches used to apply finite elements for the solution of differential equations, the approach is described and to be used for the solution of 1D heat problem. Other approaches are available in the literature. [1], [2], [3] and [6].

Weighted residual approach

The procedure in the Weighted residual approach is as follows:
Consider solving the differential equation $\boldsymbol{D}(\boldsymbol{u}(\boldsymbol{x})) = \boldsymbol{f}(\boldsymbol{x})$, if a solution $\psi(x)$ is assumed for this differential equation, then the residual for this equation will be $R(x)$ where

$$R(x) = \left(D(\psi(x)) - f(x)\right) \tag{1}$$

Of course, $R(x)$ will not be zero unless the assumed solution is an exact solution and it is not the case. Next, $R(x)$ is to be multiplied by a weighting function $w(x)$ and then to be integrated over the problem domain as.

$$\int_{x_0}^{x_L} [(D(\psi(x)) - f(x))] w(x) dx \qquad (2)$$

Now the assumed (approximation) solution can be taken over each element as

$$\psi(x) = \sum_{i=1}^{m} \psi_i N_i(x) \qquad (3)$$

Where $\boldsymbol{m}$ is the number of nodes in the element, $N_i(x)$ is the shape function for node i and ψ_i is the unknown at node i and then equation (2) becomes

$$\int_{x_0}^{x_L} \left[\left(D\left(\sum_{i=1}^{m} \psi_i N_i(x)\right) - f(x)\right)\right] w(x) dx \qquad (4)$$

The Weighted residual approach takes different names depending on the choice of the weighting function $w(x)$, some examples are

a) in the **Collocation method:** the weighting function is taken as $w_i(x) = \delta(x - x_i)$
b) in the Least-**Squares method**: the weighting function is taken as $w_i(x) = \dfrac{\partial R}{\partial a_i}$ where a_i are the unknowns in the assumed approximate solution.
c) in the Galerkin **method**: the weighting function is taken to be the same as the shape functions, that is $w_i(x) = N_i(x)$ where $N_i(x)$ is the shape function for node i

Galerkin approaches

The approach will be used for a specific 1D problem in this paper**.**

Now let us consider solving the following differential equation using **Galerkin** approach

The basic procedure can be summarized in the following five steps

Step 1 The differential equation is to be multiplied by a weight function $w_i(x)$ and perform the integral over the problem domain

Step 2 Integrate to reduce the order of the highest order term

Step 3 Choose type of elements and the order of shape functions

Step 4 Evaluate all integrals over each element, either analytically or numerically, to set up a system of algebraic equations in the unknowns.

Step 5 Solve the resulting system of equations.

Elements and Shape functions

In the application of finite elements as shown in Figure (1), the problem domain is divided into small pieces known as “elements” and the ends of each element represent a point known as a “node”

The elements are pieces or segments of the problem domain with points or nodes as shown. In the 1D domains the elements are lines while for 2D they can be triangular or rectangular and in the 3D cases they are prisms, tetrahedranes, pyramids, hexahedral or parallelepiped elements. The degree of shape functions is in general depends on the number of nodes in the element, as shown below, for elements with nodes only at the edges, the shape functions are linear and they are quadratic for elements having nodes both at the edges and at the mid between edge nodes and so on. 1D line elements and shape functions are shown in Figure (2), 2D rectangular elements and shape functions are shown in Figure (3). Other elements and associated shape functions can be found in the literature [5]. The shape functions are defined using local variable ξ. The local variable ξ is related to the system global variable x with the relation.

$$\xi = \frac{2}{l}\left(x - x_n\right) \qquad (5)$$

Where (l) is the element width and x_n is the distance from the problem origin to the center of the element n.

The 1D two node elements employ linear shape functions given by

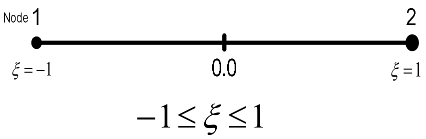
$$N_1(\xi)=\frac{1}{2}(1-\xi) \qquad\qquad N_2(\xi)=\frac{1}{2}(1+\xi)$$

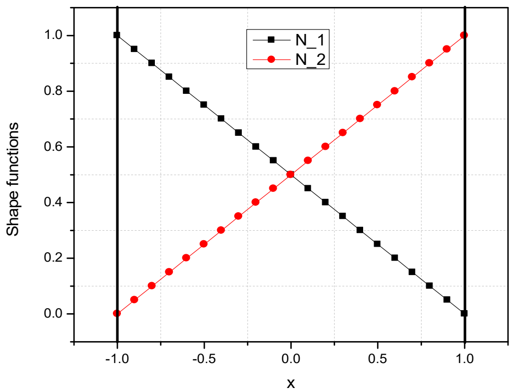
And the three node elements employ quadratic shape functions given by

$$N_1(\xi)=\frac{1}{2}\xi(1-\xi) \qquad N_2(\xi)=\left(1-\xi^2\right) \qquad N_3(\xi)=\frac{1}{2}\xi(\xi+1)$$

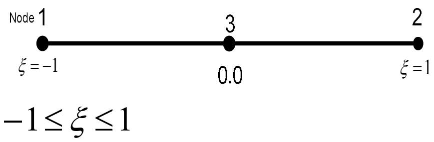
The elements and the plots of the shape functions are shown in Figure (2)

2 node 1D elements *shape functions*





<u>3 node elements</u>



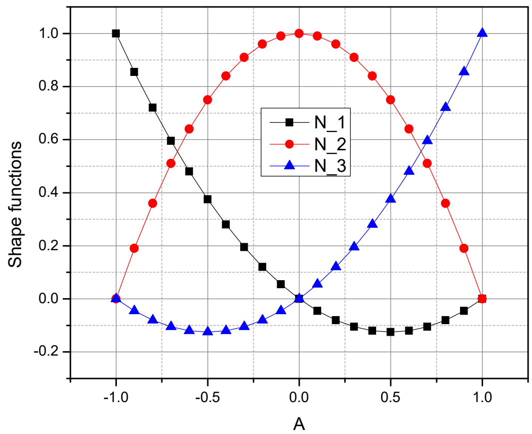


Figure (2) 1D elements and plots of the shape functions

Application of Galerkin Approach to 1D heat Problem

The steady state heat conduction in 1D with uniform heat generation is governed by the following second order differential equation.

$$k\frac{\partial^2 T}{\partial^2 x}+Q=0 \quad \text{with } 0\le x\le L \qquad (6)$$

Where $T(x)$ is the temperature function and Q is the uniform heat generation per unit volume. The finite element solution will be compared to the exact (analytical) solution which is given by

$$T(x)=\frac{Q}{2k}\left(L^2+\frac{2kL}{h}-x^2\right)+T_L \qquad (7)$$

Where:

k is the thermal conductivity of the material.

h is the convective heat transfer coefficient.

First in the finite element solutions, the equation is to be multiplied by a weighting function $w(x)$ and to be integrated over the problem domain as

$$\int_v w(x)\frac{dT}{dx}\left(k\frac{dT}{dx}\right)dv+\int_v w(x)Qdv=0$$

$$\Rightarrow \int_{x_1}^{x_2} w(x)\frac{dT}{dx}\left(k\frac{dT}{dx}\right)Adx+\int_{x_1}^{x_2} w(x)QAdx=0 \qquad (8)$$

where $dv=Adx$ and taking $w_i(x)=N_i(x)$ the same as the shape functions

Integrating the left-hand side integral of equation (8) gives

$$kAN_i(x)\frac{dT}{dx}\bigg|_{x_1}^{x_2}-kA\int_{x_1}^{x_2}\left(\frac{dN_i}{dx}\frac{dT}{dx}\right)dx+\int_{x_1}^{x_2} N_i(x)QAdx=0, \qquad i=1,2 \qquad (9)$$

Using natural (local) coordinates and T^e is taken to be the approximate solution over each element given by

$$T^e = \sum_{i=1}^{2} T_i N_i(\xi) = \begin{bmatrix} N_1(\xi) & N_2(\xi) \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \end{bmatrix} \tag{10}$$

So that

$$\begin{aligned} \frac{dT^e}{dx} &= \frac{dT^e}{d\xi}\frac{d\xi}{dx} = \frac{2}{l}\frac{dT^e}{d\xi} = \frac{2}{l}\left[\frac{dN_1(\xi)}{d\xi}T_1 + \frac{dN_2(\xi)}{d\xi}T_2\right] \\ &= \frac{1}{l}\begin{bmatrix} -1 & 1 \end{bmatrix}\begin{bmatrix} T_1 \\ T_2 \end{bmatrix} \end{aligned} \tag{11}$$

Substitutions would result into a system of equation that can be written in general as

$$\underline{\underline{M}}\,\underline{T} = \underline{G} + \underline{f} \tag{12}$$

Which can be written for any element as

$$\frac{kA}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} T_i \\ T_{i+1} \end{bmatrix} = \frac{QAl}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \underline{f} \tag{13}$$

Where $\underline{f}$ depends on the problem conditions

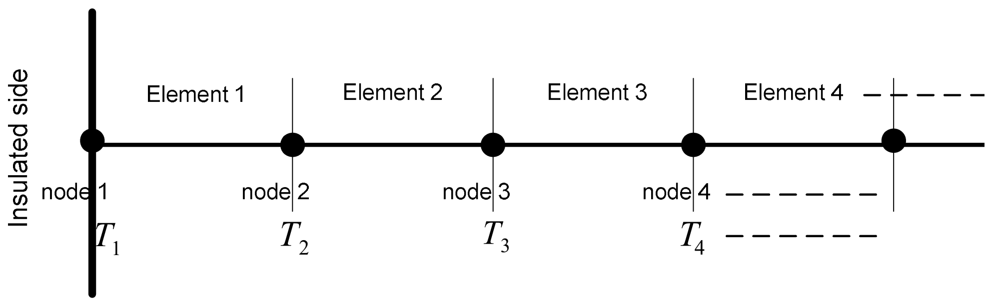


figure (3) Global system elements

For the global system as shown in figure (3), equation (13) becomes

$$\frac{kA}{l}\begin{bmatrix} 1 & -1 & & & & & & \\ -1 & 2 & -1 & & & & & \\ & -1 & 2 & -1 & & & & \\ & & -1 & 2 & -1 & & & \\ & & & -1 & 2 & -1 & & \\ & & & & & \ddots & & \\ & & & & & & \ddots & \\ & & & & & & -1 & 1 \end{bmatrix}\begin{bmatrix} T_1 \\ T_2 \\ \vdots \\ T_n \end{bmatrix} = \begin{bmatrix} Ql/2 \\ Ql \\ \vdots \\ Ql \\ Ql/2 \end{bmatrix} + \underline{f} \qquad (14)$$

Now taking the length of the system domain to be $(120mm)$, the cross sectional area $(A = 1\ m^2)$, $(k = 26\ w/m\ ^oc)$, $(Q = 0.42\ Mw/m^3)$, the left hand side is insulated and the right hand side is subjected to convection at $85\ ^oc$, the heat transfer coefficient $h = 625 w/m^2\ ^oc$.

The boundary conditions used to determine the analytical solution given in equation (6) are:

- no heat loss through the insulation in the left-hand side

  $$\frac{dT}{dx} = 0 \quad at \quad x = 0$$

- Convection at the right-hand side boundary

  $$q = -k\frac{dT}{dx} = h(T - T_L) \quad at \quad x = L$$

For the finite element solution, the system domain is divided into 8 elements $(l = 15mm)$, then the matrix in equation (14) will be

.

$$\begin{bmatrix} 1733.3 & -1733.3 & & & & & & \\ -1733.3 & 3466.6 & -1733.3 & & & & & \\ & -1733.3 & 3466.6 & -1733.3 & & & & \\ & & -1733.3 & 3466.6 & -1733.3 & & & \\ & & & -1733.3 & 3466.6 & -1733.3 & & \\ & & & & & \ddots & & \\ & & & & & & 3466.6 & -1733.3 \\ & & & & & & -1733.3 & 2358.3 \end{bmatrix}\begin{bmatrix} T_1 \\ T_2 \\ \vdots \\ T_9 \end{bmatrix} = \begin{bmatrix} 3150.0 \\ 6300.0 \\ 1950.0 \\ \vdots \\ 1950.0 \\ 56275.0 \end{bmatrix} \qquad (15)$$

As indicated in equation (15) it is clear that the element matrices are:

$$\underline{\underline{M}}^{e} = \begin{bmatrix} 1733.3 & -1733.3 \\ -1733.3 & 1733.3 \end{bmatrix}, \ \underline{\underline{G}}^{e} = \begin{bmatrix} 3150 \\ 3150 \end{bmatrix}$$

Note that except for the last element the element matrices $\underline{\underline{M}}^{e}$, $\underline{G}^{e}$ are the same and for the first 7 elements and $\underline{f} = \underline{0}$, but for the last element $\underline{f} \neq \underline{0}$, that is

$$\underline{\underline{M}}^{8} = \begin{bmatrix} 1733.3 & -1733.3 \\ -1733.3 & 1733.3 \end{bmatrix} + h\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1733.3 & -1733.3 \\ -1733.3 & 2358.3 \end{bmatrix} \quad \text{And}$$

$$\underline{f}^{8} = \underline{\begin{bmatrix} 3150 \\ 3150 \end{bmatrix}} + hAT_{w}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3150 \\ 56275 \end{bmatrix}$$

This system of algebraic equations is being solved using Gauss elimination, the obtained solution is shown along with analytical solution in figure (4).

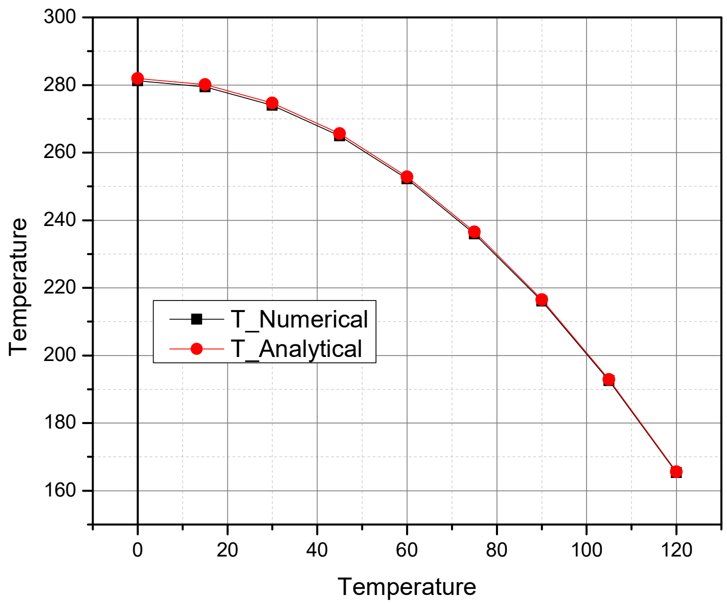


Figure (4) Numerical and Analytical Solutions

Conclusions

In the illustrating problem in the application of Galerkin approach to 1D heat problem, Finite element method is a numerical technique that produces approximate solutions in general in a very good agreement with exact solutions. The capability of the method to solve problems with complex geometries makes it one of the important numerical tools for applications in general engineering problems.

The discussions and procedures can be easily extended into problems of 2D and 3D geometries.

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